Key Differences And Similarities Between The Infinite Horizon And Overlapping Generations Models In Terms Of Assumptions, Structural Features And Overall Implications

By

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In this exercise, the Ramsey-Cass-Koopmans (RCK) and the Diamond models as explicated by Romer (1996) will be the basis of the infinite horizon and overlapping generations models, respectively. The RCK model is the synthesis of ideas from Frank Ramsey (1928), David Cass (1965) and Tjalling Koopmans (1965), which examined the effect of an endogenous savings rate in a growth model where households are infinitely lived. Peter A. Diamond (1965) investigated the effect of the national debt on economic growth by developing the overlapping generations model, which he contrasted against the infinite horizon model. These two models are similar and yet different in its assumptions, features and implications.

Both the RCK and the Diamond models rested upon basic neo-classical assumptions of perfectly competitive markets where firms maximize profits and individuals (households) maximize utility. In addition, to these models further assumed the following assumptions to be baseline:

- A production function with Capital (K), Knowledge (A) and Labour (L); F(K, AL)[1] with constant returns to scale.
- Capital is endogenous while Knowledge and Labour are exogenous
- Capital and Output are the same commodity so Capital can be consumed
- No depreciation
- Households owned the firms and therefore earned profits
- Savings and Consumption are endogenous

The two models differ in the following assumptions.

RCK Model	Diamond Model
❑ Continuous time ❑ Fixed number of infinitely lived households	❑ Discrete time ❑ Turnover in the population with individuals being born and dying continually

[1] Romer enhanced the RCK model by including the variable for Knowledge (A). The original models developed by Ramsey, Cass and Koopmans did not include this variable. I have however included this variable in this paper to maintain consistency with Romer's exposition.

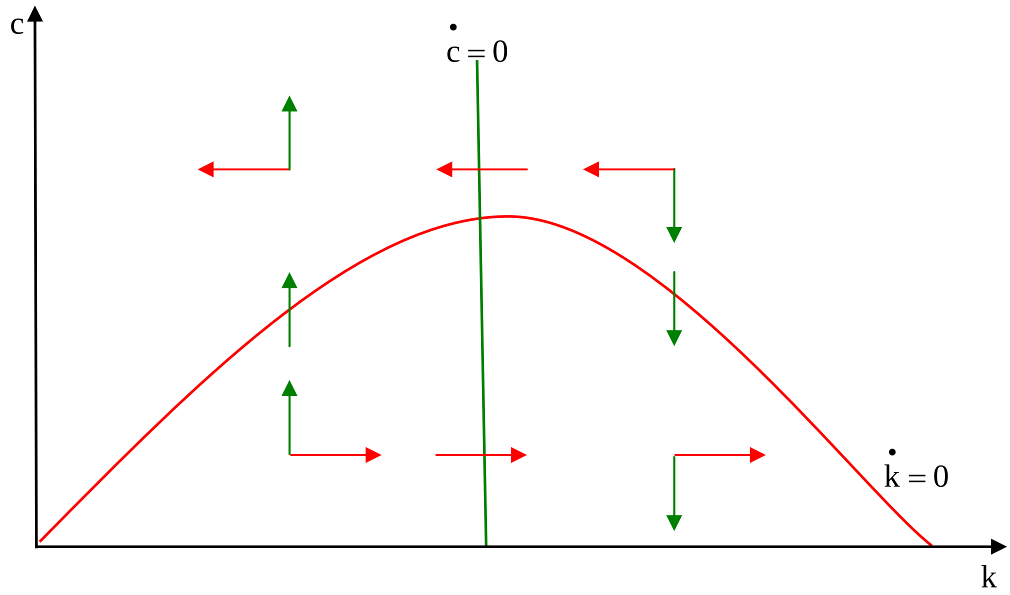
❑ Individuals continuously work and consume at all points in time.	❑ Individuals in this economy live 2 periods, where they work and consume in the earlier period and only consume in the later period. This is the reason why the model is known as the overlapping generations model.

The structural features of the two models are different because of the departure from the baseline assumptions. The different features are as follows. The derivations of the following equations are included in the Appendix.

RCK Model	**Diamond Model**
❑ Consumption Function (Budget Constraint) $\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt = k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$ $R(t) =$ Interest Rate as a function of time, $c(t) =$ Consumption per Effective Labour as a function of time, $w(t) =$ Wages per Effective Labour as a function of time, n = growth rate of Labour g = growth rate fo Knowledge k = Capital per effective labour $k(0) =$ Endowment	❑ Consumption Function (Budget Constraint) $C_t + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$ $C_t =$ Consumption $A_t =$ Knowledge $r_t =$ Interest Rate $w_t =$ wages per effective labour
❑ Utility Function $U(t) = B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$ where $B = \frac{A(0)^{1-\theta} L(0)}{H}$, $\beta = -(\rho - n - g(1-\theta))$ H=Number of Households In The Economy L=Number of workers	❑ Utility Function $U_t = \frac{C_t^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta}$ $\theta =$ Constant Relative Risk Aversion Factor $\rho =$ Preference for Present Consumption Factor
❑ Euler Equation $\frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}$	❑ Euler Equation $\frac{C_{2t+1}}{C_t} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$

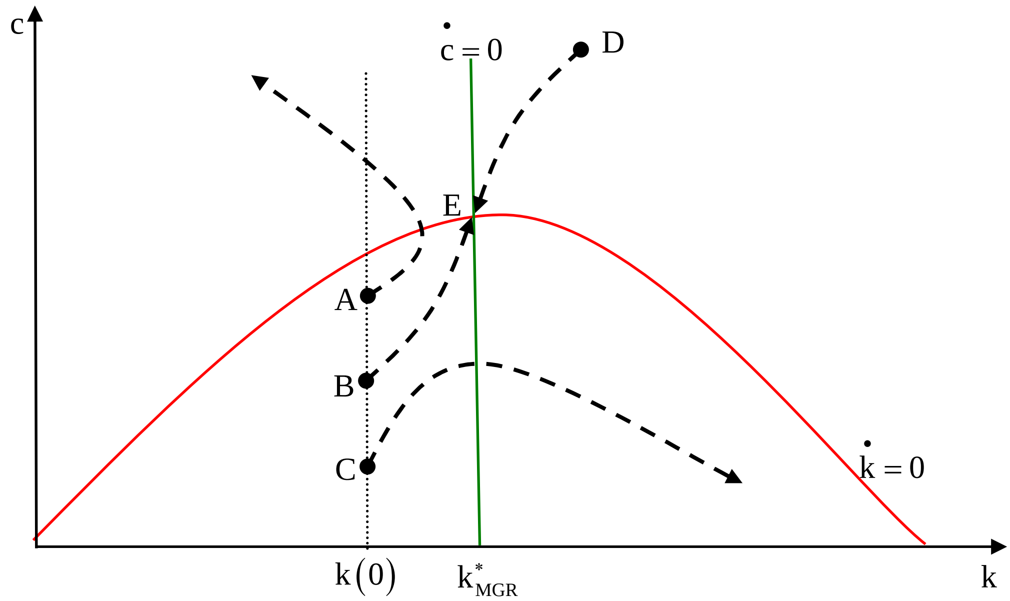
❑ Evolution path of k $\dot{k} = sf(k) - (n+g)k$ $s =$ savings rate	❑ Evolution path of k $k_{t+1} = Dk_t^{\alpha} = f(k_t)$

Graph 1 and 2 represents the dynamics of the RCK model. In the first graph, the red line maps the coordinates of c, which satisfies the condition $\dot{k} = 0$ because $sf(k) = (n+g)k$. k is stationary at all points on this line. Above this line, $\dot{k} < 0$ because $sf(k) < (n+g)k$ and hence k is decreasing which is shown by the red arrows pointing to the left. Below this line, $\dot{k} > 0$ because $sf(k) > (n+g)k$ and hence k is increasing which is shown by the red arrows pointing to the right. The vertical green line, is the locus of c and k, which meets the condition where $\dot{c} = 0$. c is stationary at all points on this line. To the left of this line, $\dot{c} > 0$ because $r(t) > \rho$ and hence c is increasing and is shown by the green arrows pointing upwards. To the right of this line, $\dot{c} < 0$ because $r(t) < \rho$ and hence c is decreasing and is shown by the green arrows pointing downwards. The equilibrium point E is the intersection of the red and green lines which satisfies the condition where $\dot{k} = 0$ and $\dot{c} = 0$. This equilibrium point is unique and stable. However, this equilibrium point is located to the left of the apex of the red line where c is at the maximum. Therefore, the equilibrium is not at the point where c is at maximum. This is because of the restriction where $\rho - n - g(1-\theta) > 0$ giving rise to $\rho > n$ when g=0 is assumed. This is at the location along the f(k) curve where it is still rising before the tangent is parallel to the breakeven investment line, $f'(k) = (n+g)k$. Intuitively, this condition reflects the behavior where individuals have an overriding preference for present consumption despite this preference being myopic and does not maximize total consumption. Hence, this equilibrium is not the true Golden Rule but the Modified Golden Rule level k^*_{MGR}.



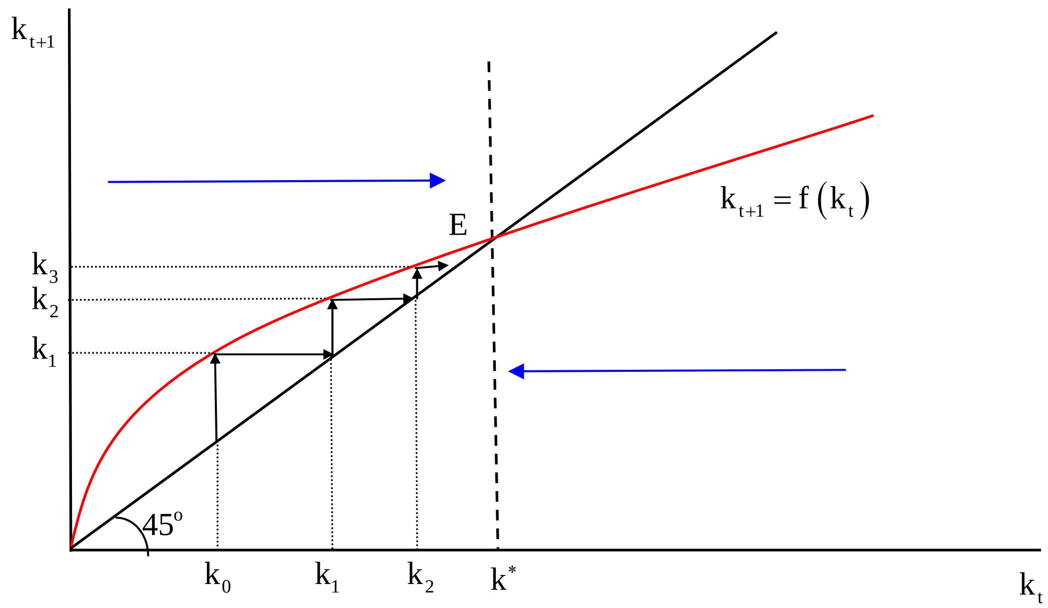
Graph 1: The Dynamics of The RCK Model

The second graph, illustrates the evolution path of c and k in the RCK model given the dynamics of the model. Only points B and D converge smoothly to the equilibrium E at the intersection of the lines. Other points are not feasible because the dynamics of this model will generate evolution paths that are not feasible given the conditions where $k_t > 0$ and $\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)} dt = k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)} dt$. Point A generates a divergent path, which violates the condition where k must be positive. The path starting from Point B is also divergent and violates the wealth is not sufficiently consumed leading to increasing wealth and reducing consumption. The reader can also work out paths for other points, which will be divergent and therefore not feasible. Only Points B and D will generate paths that will converge smoothly to the Modified Golden Rule equilibrium E. However, a pertinent point to note for the RCK model is that the starting point in the economy is critical to ensure convergence. Its dynamics requires that the starting point to exactly on the saddle path denoted by points B and D in Graph 2. Any other points and hence combinations of c and k off the saddle path lead to a divergence instead of a convergence.



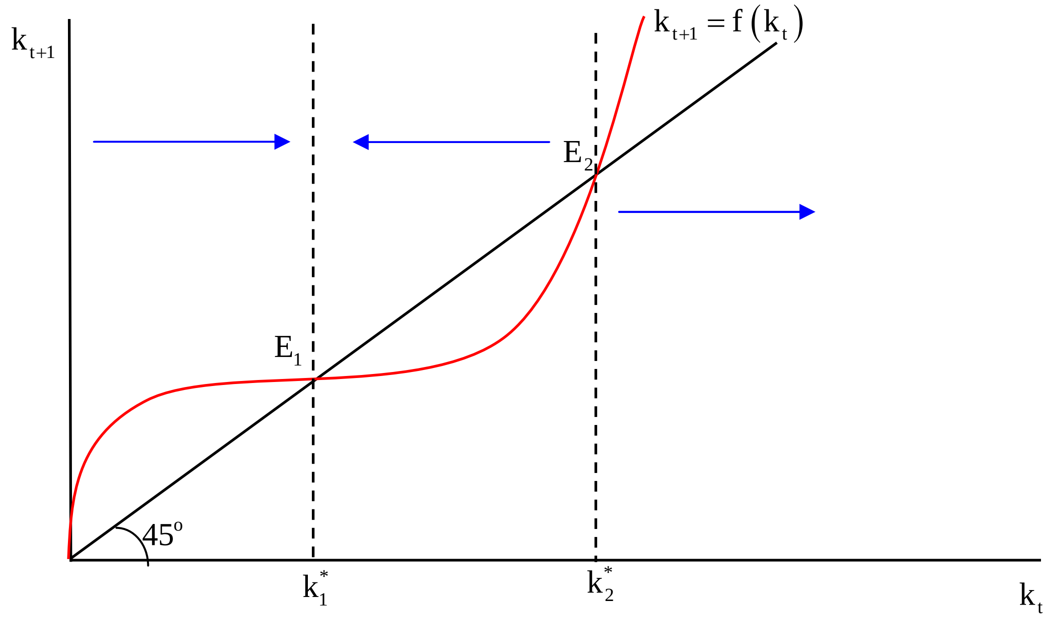
Graph 2: The RCK Model Golden Rule Equilibrium

The dynamics of the Diamond model differs from the RCK model solely because it assumes an economy with overlapping generations instead of an infinitely lived household. Graph 3 represents the dynamics of the Diamond model given the Cobb-Douglas production function. The dynamics are determined by the Euler equation $k_{t+1} = Dk_t^{\alpha} = f(k_t)$. The equilibrium point E is unique and stable. It is located where the $k_{t+1} = Dk_t^{\alpha} = f(k_t)$ locus intersects the 45^o line. At this equilibrium point, $k^* = k_t = k_{t+1}$. To the left of the vertical k^* line, $k_t < k_{t+1}$ and therefore k_t is rising in k_{t+1} until it converges to the equilibrium in the direction of the right pointing arrow. To the right of the vertical k^* line, $k_t > k_{t+1}$ and therefore k_t is decreasing in k_{t+1} until it converges to the equilibrium in the direction of the left pointing arrow. As an illustration, k_0 is to the left of k^* where $k_t < k_{t+1}$. Therefore, households will adjust the allocation of k in t and t+1 periods until it reaches in the balance. The direction of the adjustment is to the right of k_0 by increasing k in every t+1 period until equilibrium point E is reached. At the equilibrium point E, k will be stationary.

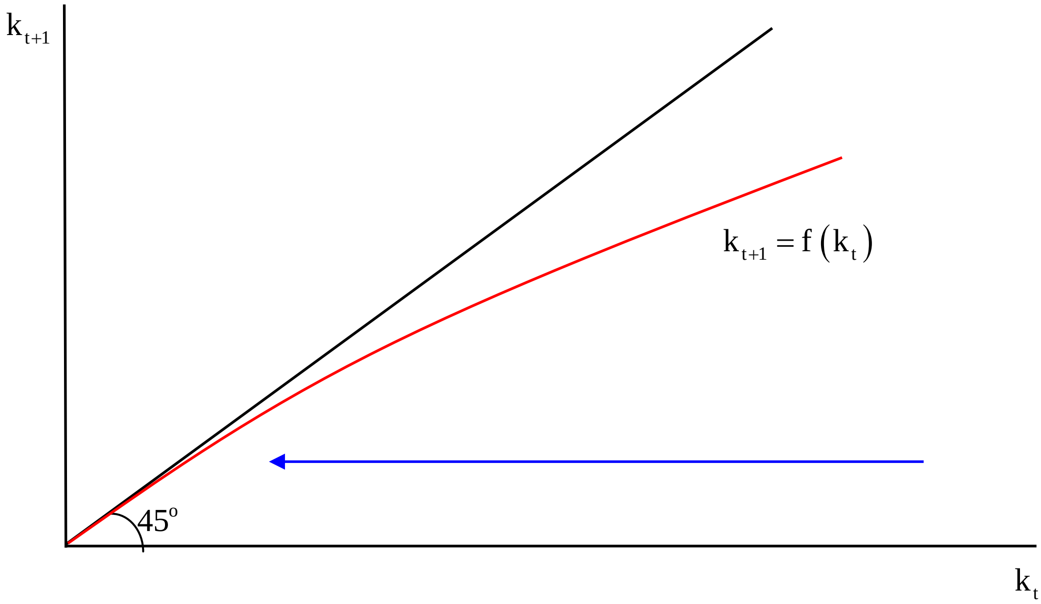


Graph 3: The Dynamics of The Diamond Model

However, the dynamics of convergence for this model depends on the shape of the k_{t+1} function. Consider the following Graph 4 and 5, which assumes non Cobb-Douglas Production Functions. The economy in Graph 4 will smoothly converge to equilibrium point E_1 at k_1^* when it starts with the condition $k_t < k_1^*$ or $k_1^* < k_t < k_2^*$. However, it will diverge if it starts with $k_t > k_2^*$. There are 2 equilibrium points E_1 and E_2. However, only E_1 is stable while E_2 is a knife-edge and unstable. The economy in Graph 5 will converge to the origin 0 regardless of the starting point.



Graph 4: The Diamond Model (Non Cobb-Douglas Production Function) A



Graph 5: The Diamond Model (Non Cobb-Douglas Production Function) B

The RCK and Diamond models will exhibit similar characteristics upon reaching their equilibrium points. At the equilibrium points, the economy will be on a balanced growth path, where k and output per effective labour will grow at the rate of technical progress g while GDP will grow at the combined labour and knowledge growth rates (n+g). The savings and consumption rates as a proportion of income will also remain constant.

The implications of the RCK and the Diamond models are different. Given the RCK model following the saddle path, the Modified Golden Rule level is consistently attained within a competitive market. This equilibrium is Pareto Optimum and cannot be improved without making another household worse off. To illustrate, the Government by forcing and increase in the savings rate may lead to a divergence where consumption reduces while wealth increases ad infinitum. Such may be the case when the Government forces higher savings through higher EPF contributions. Therefore, any Government action to affect savings and consumption will adversely affect welfare.

However, the Diamond model does not necessarily converge to the Golden Rule level. If it converges to the Golden Rule level, it is by fortune and nothing else. Equilibrium levels that are not the Golden Rule are not Pareto Optimum and causes dynamic inefficiencies because rearranging inter-temporal distribution of consumption can improve welfare. One example of dynamic inefficiency in this model is when the present generation borrows to consume and passes on the burden of debt to the future generation.

In the Diamond model, the Government needs to force higher savings to shift the equilibrium towards the Golden Rule level, if the existing equilibrium level is at a point savings rate is below the optimal Golden Rule level. At the other end is the equilibrium situation when households are saving too much and hence over investing beyond the optimal Golden Rule level. The Government needs to remedy this situation by selling bonds because this action shifts the equilibrium towards the optimal Golden Rule level.

Another implication will be in the example of an increase in Government spending. The RCK model does not differentiate between the means for the Government to finance its spending. The household is indifferent regardless if the means are through selling bonds or raising taxes. If the Government raises tax, then households will borrow against future income to smoothen the ripple. However, if Government sells bonds instead, then households will save the additional wealth from the bond to pay for future increase in taxes that the government will raise in order to redeem the bonds. Therefore, regardless of the means, households will adjust accordingly so that their present consumption does not change and the economy will continue to maintain the Pareto Optimum Modified Golden rule level equilibrium. This implication is important because it means that Government spending might not have any desired effect on economic growth.

The Diamond model brings about a contrasting implication because of inter-generation relationship in the instance where Government increases spending. If the Government raises tax to finance its spending, the present generation may be worse off through reduced income if at the margin, the loss of income through higher tax is larger that the additional income through increased government spending. However, if the Government sells bonds instead, the present generation may consume the additional wealth, while leaving the future generation to bear the burden of higher taxes, which the government will need to raise to retire the bonds. This behavior may seem selfish because it raises the welfare of the present generation to the detriment of future generations. The future generations are impoverished by the callousness of the present generation. This means that in the Diamond model, Government's choice of raising tax or selling bonds to finance its spending affects households. Households perceive increasing wealth when Government increases spending through selling bonds, which lead to greater consumption. Therefore, in this manner Government spending will have the desired effect on the economy.

Appendix

The Derivation for the RCK model

a) Consumption Function (Budget Constraint)

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt = k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$$

$R(t) =$ Interest Rate as a function of time,

$c(t) =$ Consumption per Effective Labour as a function of time,

$w(t) =$ Wages per Effective Labour as a function of time,

n = growth rate of Labour, g = growth rate fo Knowledge

k = Capital per effective labour, $k(0) =$ Endowment

b) Utility Function

$$U(t) = B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

where $B = \dfrac{A(0)^{1-\theta} L(0)}{H}$, $\beta = -(\rho - n - g(1-\theta))$

H=Number of Households In The Economy, L=Number of workers

c) Optimisation

$$L = B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt - \lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} \left[w(t) - c(t)\right] e^{(n+g)t} dt\right]$$

$$L' = Be^{-\beta t} \dot{c}(t)^{-\theta} = \lambda e^{-R(t)} e^{(n+g)t}$$

$$\Rightarrow \ln B - \beta t - \theta \ln \dot{c}(t) = \ln \lambda - R(t) + (n+g)t$$

Differenciate both Sides wrt t,

$$\Rightarrow -\beta - \theta \frac{\dot{c}(t)}{c(t)} = -r(t) + n + g \Rightarrow \frac{\dot{c}(t)}{c(t)} = \frac{r(t) + n + g - \beta}{\theta}$$

$$\Rightarrow \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - n - g - (\rho - n - g(1-\theta))}{\theta}$$

$$\Rightarrow \frac{\dot{c}(t)}{c(t)} = \frac{r(t) + n + g - \rho + n + g - \theta g}{\theta} = \frac{r(t) - \rho - \theta g}{\theta}$$

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(t)}{c(t)} + \frac{\dot{A}(t)}{A(t)} = \frac{r(t) - \rho - \theta g}{\theta} + g = \frac{r(t) - \rho}{\theta}$$

Therefore, the Euler equation is

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}$$

The Derivation for the Diamond model

a) Consumption Function (Budget Constraint)

$$C_t + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$$

C_t = Consumption, A_t = Knowledge, r_t = Interest Rate, w_t = wages per effective labour

b) Utility Function

$$U_t = \frac{C_t^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

θ = Constant Relative Risk Aversion Factor, ρ = Preference for Present Consumption Factor

c) Optimisation

$$L = \frac{C_t^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda \left[A_t w_t - C_t - \frac{1}{1+r_{t+1}} C_{2t+1} \right]$$

$$\frac{dL}{dC_t} = C_t^{-\theta} = \lambda$$

$$\frac{dL}{dC_{2t+1}} = \frac{1}{1+\rho} C_{2t+1}^{-\theta} = \lambda \frac{1}{1+r_{t+1}} \Rightarrow \frac{1}{1+\rho} C_{2t+1}^{-\theta} = C_t^{-\theta} \frac{1}{1+r_{t+1}}$$

$$\Rightarrow \frac{C_{2t+1}}{C_t} = \left(\frac{1+r_{t+1}}{1+\rho} \right)^{1/\theta}$$

d) Evolution path of k

From previous, $\frac{C_{2t+1}}{C_t} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} \Rightarrow C_{2t+1} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} C_t$

$$C_t + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t \Rightarrow C_t + \frac{1}{1+r_{t+1}} \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} C_t = A_t w_t$$

$$\Rightarrow C_t \left[1 + \frac{1}{1+r_{t+1}} \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}\right] = A_t w_t \Rightarrow C_t = \frac{1}{\left[1 + \frac{1}{1+r_{t+1}} \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}\right]} A_t w_t$$

savings rate s_t

$$\Rightarrow s_t = \left[1 - \frac{1}{\left[1 + \frac{1}{1+r_{t+1}} \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}\right]}\right] = \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{(1-\theta)} + (1+r)^{(1-\theta)/\theta}}$$

Assume, θ=1 to equalize income and substittion effect $\Rightarrow s_t = \frac{1}{2+\rho}$

Assume Cobb-Douglas function so that $f(k) = k^{\alpha}$ so that $w_t = (1-\alpha) k_t^{\alpha}$

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s_t w_t = \frac{1}{(1+n)(1+g)(2+\rho)} (1-\alpha) k_t^{\alpha} = D k_t^{\alpha} = f(k_t)$$

$$\therefore \; k_{t+1} = D k_t^{\alpha} = f(k_t)$$

References

Cass, David (1965), *Optimum Growth in An Aggregative Model of Capital Accumulation, Review of Economic Studies 32*

Diamond, Peter (1965), *National debt in a Neocalssical Growth Model, AER 55*

Koopmans, Tjalling (1965), *On The Concept of Optimal Growth in The Economic Approach to Development Planning*

Ramsey, Frank (1928), *A Mathematical Theory of Saving, Economic Journal 38*

Romer, David (1996), *Advanced Macroeconomics*